

CURVATURE HOMOGENEOUS SPACELIKE JORDAN OSSERMAN PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. Let $s \geq 2$. We construct Ricci flat pseudo-Riemannian manifolds of signature $(2s, s)$ which are not locally homogeneous but whose curvature tensors never the less exhibit a number of important symmetry properties. They are curvature homogeneous; their curvature tensor is modeled on that of a local symmetric space. They are spacelike Jordan Osserman with a Jacobi operator which is nilpotent of order 3; they are not timelike Jordan Osserman. They are k -spacelike higher order Jordan Osserman for $2 \leq k \leq s$; they are k -timelike higher order Jordan Osserman for $s + 2 \leq k \leq 2s$, and they are not k -timelike higher order Jordan Osserman for $2 \leq s \leq s + 1$.

1. INTRODUCTION

Let ∇ be the Levi-Civita connection of a pseudo-Riemannian manifold (M, g) of signature (p, q) and dimension $m = p + q$. Let

$$R(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x, y]} \quad \text{and} \\ R(x, y, z, w) := g(R(x, y)z, w)$$

be the associated Riemann curvature operator and curvature tensor. Manifolds whose Riemann curvature has a high degree of symmetry are important in many contexts. Usually this symmetry arises from an underlying symmetry of the metric tensor. One says that (M, g) is *locally homogeneous* if given any two points P and Q of M , there exists a local isometry ψ from some neighborhood U_P of P to some neighborhood U_Q of Q such that $\psi P = Q$. One says that (M, g) is a *local symmetric space* when $\nabla R = 0$; local symmetric spaces are always locally homogeneous.

In this note, we shall exhibit a family of manifolds whose curvature tensor has a high degree of symmetry in several different senses, but which are not locally homogeneous. We begin by reviewing some definitions:

1.1. Curvature homogeneous manifolds. The manifold (M, g) is said to be *curvature homogeneous* if given any two points $P, Q \in M$, there is a linear isomorphism $\Psi : T_P M \rightarrow T_Q M$ so that $\Psi^* g_Q = g_P$ and so that $\Psi^* R_Q = R_P$; see Kowalski, Tricerri, and Vanhecke [31, 32] for further details. If (M, g) is curvature homogeneous, then the curvature tensor looks the same for every point of M .

There is a useful equivalent characterization of curvature homogeneity. Consider the triple $\mathcal{V} := (V, g_V, R_V)$ where g_V is a non-degenerate inner product of signature (p, q) on a real vector space V of dimension $m := p + q$ and R_V is an algebraic curvature tensor on V ; i.e. a 4 tensor which satisfies the usual symmetries of the Riemann curvature tensor:

$$(1.a) \quad R_V(x, y, z, w) = R_V(z, w, x, y) = -R_V(y, x, z, w), \quad \text{and} \\ (1.b) \quad R_V(x, y, z, w) + R_V(y, z, x, w) + R_V(z, x, y, w) = 0.$$

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Equation (1.a) gives \mathbb{Z}_2 symmetries; Equation (1.b) is the *first Bianchi identity*. We say that \mathcal{V} is a *model space* for (M, g, R) if given any point $P \in M$, there exists a linear isomorphism $\Psi : T_P M \rightarrow V$ so $\Psi^* g_V = g_P$ and so $\Psi^* R_V = R_P$; (M, g) is curvature homogeneous if and only if there exists a model space for (M, g, R) .

1.2. The Jacobi operator. If x is a tangent vector at a point P of M , then the *Jacobi operator* $J(x) : y \rightarrow R(y, x)x$ is a self-adjoint endomorphism of the tangent space $T_P M$. We say that (M, g) is *spacelike* (resp. *timelike*) *Jordan Osserman* if the Jordan normal form of J is constant on the bundle of unit spacelike (resp. timelike) tangent vectors.

In the Riemannian setting ($p = 0$), work of Chi [14] and Nikolayevsky [33, 34] shows that if $m \neq 8, 16$, then any spacelike Jordan Osserman manifold is a 2 point homogeneous space; this settles in the affirmative for these dimensions a question raised by Osserman [36]. In the Lorentzian setting ($p = 1$), any spacelike or timelike Jordan Osserman manifold necessarily has constant sectional curvature [1, 18]. In the higher signature setting ($p > 1, q > 1$) the situation is far from clear; we refer to [2, 3, 5, 9, 17, 20, 22] for some partial results.

1.3. The higher order Jacobi operator. Stanilov and Videv [37] constructed a higher order Jacobi operator. Let $\{e_1, \dots, e_r\}$ be an orthonormal basis for a spacelike (resp. timelike) r -plane π in the tangent bundle. The *higher order Jacobi operator* $J(\pi) := J(e_1) + \dots + J(e_r)$ does not depend on the particular orthonormal basis chosen. One says that (M, g) is *k-spacelike* (resp. *k-timelike*) *higher order Jordan Osserman* if the Jordan normal form of $J(\cdot)$ is constant on the Grassmannian of unoriented spacelike (resp. timelike) k -planes. As setting $k = 1$ recovers the previous setting, we shall assume $k \geq 2$. The k -spacelike higher order Jordan Osserman manifolds have been classified in the Riemannian setting [21] and in the Lorentzian setting [26] but little is known in the higher signature setting apart from some examples given in [23].

1.4. Curvature homogeneous manifolds which are not locally homogeneous. It is clear that locally homogeneous manifolds are curvature homogeneous. The somewhat surprising fact is that the converse fails – there are curvature homogeneous manifolds which are **not** locally homogeneous. For a further discussion, we refer to [6, 7, 29, 39, 40, 42] for Riemannian manifolds, to [10, 11, 12, 13, 28] for Lorentzian manifolds, and to [30, 35] for affine manifolds.

In the higher signature setting, there are relatively few examples known of curvature homogeneous manifolds which are not homogeneous. The case of signature (2,2) has been studied extensively [4, 8, 15, 19]; we also refer to [27] for results concerning isoperimetric hypersurfaces. More generally, let $p \geq 2$. It is known that [16, 23, 24] there are pseudo-Riemannian manifolds of neutral signature (p, p) which are curvature homogeneous but which are not locally homogeneous. These manifolds are spacelike and timelike Jordan Osserman. Thus the curvature tensors of these manifolds exhibit a high degree of symmetry. The Jacobi operator of these manifolds is nilpotent of order 2.

1.5. Manifolds of signature $(2s, s)$. Let $s \geq 2$. In this paper, we shall extend previous work [25] to create **new** examples of pseudo-Riemannian manifolds of signature $(2s, s)$ whose Riemann curvature tensor also has a high degree of symmetry. The manifolds in this family are all curvature homogeneous with curvature tensor modeled on that of a symmetric space. Generic members of the family are not locally homogeneous. They are spacelike Jordan Osserman but not timelike Jordan Osserman. They are k -spacelike higher order Jordan Osserman for $2 \leq k \leq s$; they are k -timelike higher order Jordan Osserman if and only if $s + 2 \leq k \leq 2s$. Their Jacobi operators are nilpotent of order 3.

Fix $s \geq 2$. We define the pseudo-Riemannian manifolds that we shall be studying and their associated curvature model as follows:

Definition 1.1. Let $\vec{u} := (u_1, \dots, u_s)$, $\vec{t} := (t_1, \dots, t_s)$, and $\vec{v} := (v_1, \dots, v_s)$ give coordinates $(\vec{u}, \vec{t}, \vec{v})$ on \mathbb{R}^{3s} for $s \geq 2$. Let

$$F(\vec{u}) := f_1(u_1) + \dots + f_s(u_s)$$

be a smooth function on an open subset $\mathcal{O} \subset \mathbb{R}^s$. Let

$$|u|^2 := \sum_{1 \leq i \leq s} u_i^2 \quad \text{and} \quad u \cdot t := \sum_{1 \leq i \leq s} u_i t_i.$$

Define a pseudo-Riemannian metric g_F of signature $(2s, s)$ on $M_F := \mathcal{O} \times \mathbb{R}^{2s}$ whose non-zero components are given by:

$$\begin{aligned} g_F(\partial_i^u, \partial_i^u) &= -2F(\vec{u}) - 2u \cdot t, \\ g_F(\partial_i^u, \partial_i^v) &= g_F(\partial_i^v, \partial_i^u) = 1, \\ g_F(\partial_i^t, \partial_i^t) &= -1. \end{aligned}$$

Set $F_{/j} := \partial_j^u F = \partial_j^u f_j$, $F_{/ij} := \partial_j^u F_{/i}$, etc. Note that $F_{/ij} = 0$ for $i \neq j$. Define

$$(1.c) \quad \alpha_F := \sum_{1 \leq i \leq s} \{F_{/iii} + 4u_i\}^2.$$

Definition 1.2. Let $\{U_1, \dots, U_s, T_1, \dots, T_s, V_1, \dots, V_s\}$ be a basis for \mathbb{R}^{3s} where $s \geq 2$. Let $\mathcal{V}_{3s} := (\mathbb{R}^{3s}, g_{3s}, R_{3s})$ where the non-zero entries of the metric g_{3s} and of the algebraic curvature tensor R_{3s} , up to the \mathbb{Z}_2 symmetries of Equation (1.a), are

$$(1.d) \quad \begin{aligned} g_{3s}(U_i, V_i) &= g_{3s}(V_i, U_i) = 1, \quad g_{3s}(T_i, T_i) = -1, \quad \text{and} \\ R_{3s}(U_i, U_j, U_j, T_i) &= 1 \quad \text{for} \quad i \neq j. \end{aligned}$$

Set $Z_i^\pm := U_i \pm \frac{1}{2}V_i$. Then $\text{Span}\{Z_i^+\}$ is a maximal spacelike subspace of \mathbb{R}^{3s} and $\text{Span}\{T_i, Z_i^-\}$ is the complementary maximal timelike subspace. Thus \mathbb{R}^{3s} has signature $(2s, s)$. A basis $\mathcal{B} = \{\tilde{U}_1, \dots, \tilde{U}_s, \tilde{T}_1, \dots, \tilde{T}_s, \tilde{V}_1, \dots, \tilde{V}_s\}$ for \mathbb{R}^{3s} is said to be *normalized* if the relations given above in Display (1.d) hold for \mathcal{B} .

Theorem 1.3. Let $s \geq 2$. The manifold (M_F, g_F) is a pseudo-Riemannian manifold of signature $(2s, s)$ which is Ricci flat. We have:

- (1) (M_F, g_F) is curvature homogeneous with model \mathcal{V}_{3s} .
- (2) (M_F, g_F) is spacelike Jordan Osserman but not timelike Jordan Osserman.
- (3) (M_F, g_F) is k -spacelike higher order Jordan Osserman for $2 \leq k \leq s$; (M_F, g_F) is k -timelike higher order Jordan Osserman if and only if $s + 2 \leq k \leq 2s$.
- (4) If there exists a local isometry ψ of (M_F, g_F) with $\psi(P) = Q$, then one has $\alpha_F(P) = \alpha_F(Q)$. Thus (M_F, g_F) is not locally homogeneous for generic F .

1.6. Outline of the paper. In Section 2, we establish Assertion (1) of Theorem 1.3 by determining R and ∇R for (M_F, g_F) . It will follow that α_F vanishes identically if and only if (M_F, g_F) is a local symmetric space. By choosing F suitably, one sees then that \mathcal{V}_{3s} is the model for the curvature tensor of a Ricci flat local symmetric space. In Sections 3 and 4, we prove Assertions (2) and (3) of Theorem 1.3 by establishing the corresponding results for the model space \mathcal{V}_{3s} . In Section 5, we complete the proof of Theorem 1.3 by constructing additional natural structures on the manifold (M_F, g_F) that show α_F is preserved by local isometries.

2. THE CURVATURE TENSOR OF THE MANIFOLD (M_F, g_F)

We begin our study of the manifold (M_F, g_F) by showing:

Lemma 2.1. Let R_F and ∇R_F be the curvature tensor and the covariant derivative curvature tensor of the pseudo-Riemannian manifold (M_F, g_F) defined above. Then:

- (1) The non-zero entries in R_F and ∇R_F are, up to the usual \mathbb{Z}_2 symmetries,
 - (a) $R_F(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u) = F_{/ii} + F_{/jj} + |u|^2$.

- (b) $R_F(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^t) = 1$.
- (c) $\nabla R_F(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u; \partial_i^u) = F_{/iii} + 4u_i$.
- (2) If $\{z_1, \dots, z_6\}$ are tangent vectors, then $R_F(z_1, z_2)R_F(z_3, z_4)R_F(z_5, z_6) = 0$.
- (3) If z is a tangent vector, then $J_F(z)^3 = 0$.
- (4) The manifold (M_F, g_F) is Ricci flat.

Proof. Let $i \neq j$. The non-zero Christoffel symbols of the second kind are given by:

$$\begin{aligned} g_F(\nabla_{\partial_i^u} \partial_i^u, \partial_i^u) &= -F_{/i} - t_i, \\ g_F(\nabla_{\partial_i^u} \partial_i^u, \partial_j^u) &= F_{/j} + t_j, & g_F(\nabla_{\partial_i^u} \partial_j^u, \partial_i^u) &= g_F(\nabla_{\partial_j^u} \partial_i^u, \partial_i^u) = -F_{/j} - t_j, \\ g_F(\nabla_{\partial_i^u} \partial_i^u, \partial_i^t) &= u_i, & g_F(\nabla_{\partial_i^u} \partial_i^t, \partial_i^u) &= g_F(\nabla_{\partial_i^t} \partial_i^u, \partial_i^u) = -u_i, \\ g_F(\nabla_{\partial_i^u} \partial_i^u, \partial_j^t) &= u_j, & g_F(\nabla_{\partial_i^u} \partial_j^t, \partial_i^u) &= g_F(\nabla_{\partial_j^t} \partial_i^u, \partial_i^u) = -u_j. \end{aligned}$$

We may then raise indices to see the non-zero covariant derivatives are given by:

$$\begin{aligned} \nabla_{\partial_i^u} \partial_i^u &= -(F_{/i} + t_i) \partial_i^v + \sum_{k \neq i} (F_{/k} + t_k) \partial_k^v - \sum_{1 \leq k \leq s} u_k \partial_k^t, \\ \nabla_{\partial_i^u} \partial_j^u &= -(F_{/j} + t_j) \partial_i^v - (F_{/i} + t_i) \partial_j^v, \\ \nabla_{\partial_i^u} \partial_i^t &= \nabla_{\partial_i^t} \partial_i^u = -u_i \partial_i^v, \quad \text{and} \\ \nabla_{\partial_i^u} \partial_j^t &= \nabla_{\partial_j^t} \partial_i^u = -u_j \partial_i^v. \end{aligned}$$

We have $\nabla_{\partial_i^v} = 0$. Thus if at least one $z_\mu \in \{\partial_i^v\}$, then $R_F(z_1, z_2, z_3, z_4) = 0$. Similarly, if at least two of the z_μ belong to $\{\partial_i^t\}$, then $R_F(z_1, z_2, z_3, z_4) = 0$. Finally, as $F_{/ij} = 0$ for $i \neq j$, $R_F(\partial_i^u, \partial_j^u, \partial_k^u, \star) = 0$ if the indices $\{i, j, k\}$ are distinct.

The interaction term $-\sum_{1 \leq k \leq s} u_k \partial_k^t$ in $\nabla_{\partial_i^u} \partial_i^u$ is in many ways the crucial term. We prove Assertions (1a) and (1b) by computing:

$$\nabla_{\partial_i^u} \nabla_{\partial_j^u} \partial_j^u = F_{/ii} \partial_i^v - \partial_i^t + |u|^2 \partial_i^v \quad \text{and} \quad \nabla_{\partial_j^u} \nabla_{\partial_i^u} \partial_j^u = -F_{/jj} \partial_i^v.$$

We have similarly that $\nabla R_F(X_1, X_2, X_3, X_4; X_5) = 0$ if at least one of the X_i belongs to $\text{Span}\{T_i, V_i\}$. Furthermore, up to the usual \mathbb{Z}_2 symmetries, the only non-zero component of ∇R_F is given by:

$$\begin{aligned} &\nabla R_F(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u; \partial_i^u) \\ &= \partial_i^u R_F(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u) - 2R_F(\nabla_{\partial_i^u} \partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u) - 2R_F(\partial_i^u, \nabla_{\partial_i^u} \partial_j^u, \partial_j^u, \partial_i^u) \\ &= F_{/iii} + 2u_i + 2R_F(\sum_{1 \leq k \leq s} u_k \partial_k^t, \partial_j^u, \partial_j^u, \partial_i^u) + 0 = F_{/iii} + 4u_i. \end{aligned}$$

This establishes Assertion (1c).

Assertions (2) and (3) follow from Assertion (1). Since $J_F(z)^3 = 0$, 0 is the only eigenvalue of $J_F(z)$. Thus $\rho_F(z, z) := \text{Tr}(J_F(z)) = 0$ and (M_F, g_F) is Ricci flat. \square

Proof of Theorem 1.3 (1). Fix $P \in M_F$. Let constants ε_i and ϱ_i be given. We define a new basis for $T_P M$ by setting:

$$U_i := \partial_i^u + \varepsilon_i \partial_i^t + \varrho_i \partial_i^v, \quad T_i := \partial_i^t + \varepsilon_i \partial_i^v, \quad \text{and} \quad V_i := \partial_i^v.$$

Let $i \neq j$. Since $g_F(U_i, T_i) = \varepsilon_i - \varepsilon_i = 0$, the possibly non-zero entries of g_F and R_F are, up to the usual \mathbb{Z}_2 symmetries, given by

$$\begin{aligned} g_F(U_i, U_i) &= g_F(\partial_i^u, \partial_i^u) - \varepsilon_i^2 + 2\varrho_i, \\ g_F(T_i, T_i) &= -1, \quad g_F(U_i, V_i) = 1, \\ R_F(U_i, U_j, U_j, T_i) &= 1, \quad \text{and} \\ R_F(U_i, U_j, U_j, U_i) &= F_{/ii} + F_{/jj} + |u|^2 + 2\varepsilon_i + 2\varepsilon_j. \end{aligned}$$

We set

$$\varepsilon_i := -\frac{1}{2}F_{/ii} - \frac{1}{4}|u|^2 \quad \text{and} \quad \varrho_i := \frac{1}{2}\{\varepsilon_i^2 - g_F(\partial_i^u, \partial_i^u)\}.$$

This ensures that $g_F(U_i, U_i) = 0$ and $R_F(U_i, U_j, U_j, U_i) = 0$ and establishes the existence of a basis with the normalizations of Definition 1.2. \square

Remark 2.2. Note as a useful scholium to the computations performed above that we can express the function α_F of Equation (1.c) in the form:

$$\begin{aligned}\alpha_F &= \frac{1}{4} \sum_{i,j,k,l,n} \{\nabla R_F(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x; \partial_n^x)\}^2 \\ &= \frac{1}{4} \sum_{i,j,k,l,n} \{\nabla R_F(U_i, U_j, U_k, U_l; U_n)\}^2.\end{aligned}$$

Thus ∇R_F vanishes if and only if $\alpha_F = 0$. Were one to take $f_i := -\frac{1}{6}u_i^3$, then α_F would vanish identically. Consequently, there exist local symmetric spaces in the family we are considering.

3. THE JACOBI OPERATOR OF \mathcal{V}_{3s}

In light of Theorem 1.3 (1), one sees that Assertion (2) of Theorem 1.3 will follow from the corresponding assertions for the model space \mathcal{V}_{3s} :

Lemma 3.1. *Let J_{3s} be the Jacobi operator of R_{3s} for $s \geq 2$.*

- (1) *If $g_{3s}(X, X) > 0$, $\text{Rank}\{J_{3s}(X)\} = 2(s-1)$ and $\text{Rank}\{J_{3s}(X)^2\} = s-1$.*
- (2) *If X is any element of \mathbb{R}^{3s} , $J_{3s}(X)^3 = 0$.*
- (3) *The model space \mathcal{V}_{3s} is spacelike Jordan Osserman.*
- (4) *The model space \mathcal{V}_{3s} is not timelike Jordan Osserman.*

Proof. There is an additional useful symmetry which plays a crucial role. Let

$$O(s) := \{\xi = (\xi_{ij}) : \sum_{1 \leq i \leq s} \xi_{ij} \xi_{ik} = \delta_{jk}\} \subset \mathbb{M}_s(\mathbb{R})$$

be the standard orthogonal group of $s \times s$ real matrices. Define a diagonal action of $O(s)$ on \mathbb{R}^{3s} which preserves the structures g_{3s} and R_{3s} by setting:

$$(3.a) \quad \xi : U_i \rightarrow \sum_j \xi_{ij} U_j, \quad \xi : T_i \rightarrow \sum_j \xi_{ij} T_j, \quad \text{and} \quad \xi : V_i \rightarrow \sum_j \xi_{ij} V_j.$$

Let X be a spacelike vector. By applying a symmetry of the form described in Equation (3.a), we may assume that

$$X = a_1 U_1 + \sum_{1 \leq i \leq s} \{b_i T_i + c_i V_i\} \quad \text{where} \quad 2a_1 c_1 - \sum_{1 \leq i \leq s} b_i^2 > 0.$$

Thus $a_1 \neq 0$. Let $i \geq 2$. There exist real numbers $\varepsilon_{ik} \in \mathbb{R}$, where $\varepsilon_{ik} = \varepsilon_{ik}(a, b, c)$ plays no role in the subsequent development, so that

$$\begin{aligned}J_{3s}(X) : X &\rightarrow 0, & J_{3s}(X) : T_1 &\rightarrow 0, & J_{3s}(X) : V_1 &\rightarrow 0, \\ J_{3s}(X) : U_i &\rightarrow -a_1^2 T_i - \sum_{1 \leq k \leq s} \varepsilon_{ik} V_k, & J_{3s}(X) : T_i &\rightarrow a_1^2 V_i, & J_{3s}(X) : V_i &\rightarrow 0.\end{aligned}$$

This establishes Assertion (1). Assertion (2) is immediate from the definition and Assertion (3) follows from Assertions (1) and (2). To establish Assertion (4), we note that $Z_1^- := U_1 - \frac{1}{2}V_1$ is a unit timelike vector with $J_{3s}(Z_1^-) \neq 0$. On the other hand, T_1 is also a unit timelike vector with $J_{3s}(T_1) = 0$. Thus the Jordan normal form of J_{3s} is not constant on the pseudo-sphere of unit timelike vectors. \square

4. THE HIGHER ORDER JACOBI OPERATOR OF \mathcal{V}_{3s}

We establish Assertion (3) of Theorem 1.3 by proving:

Lemma 4.1. *Let J_{3s} be the Jacobi operator of R_{3s} for $s \geq 2$.*

- (1) *If π is a spacelike k -plane for $2 \leq k \leq s$, then $\text{Rank}\{J_{3s}(\pi)\} = 2s$, $\text{Rank}\{J_{3s}(\pi)^2\} = s$, and $J_{3s}(\pi)^3 = 0$.*
- (2) *The model space \mathcal{V}_{3s} is k -spacelike higher order Jordan Osserman for $2 \leq k \leq s$.*
- (3) *If π is a timelike k -plane for $s+2 \leq k \leq 2s$, then $\text{Rank}\{J_{3s}(\pi)\} = 2s$, $\text{Rank}\{J_{3s}(\pi)^2\} = s$, and $J_{3s}(\pi)^3 = 0$.*
- (4) *The model space \mathcal{V}_{3s} is k -timelike higher order Jordan Osserman for $s+2 \leq k \leq 2s$.*
- (5) *The model space \mathcal{V}_{3s} is not k -timelike higher order Jordan Osserman if $2 \leq k \leq s+1$.*

Proof. Let $B_U := \mathbb{R}^{3s} / \text{Span}\{T_i, V_i\}$ and let σ_U be the natural projection from \mathbb{R}^{3s} to B_U . Fix a normalized basis $\mathcal{B} = \{U_i, T_i, V_i\}$ for \mathbb{R}^{3s} . Define a positive definite inner product $g_{U,\mathcal{B}}$ on $B_U = \text{Span}\{\sigma_U(U_i)\}$ so

$$g_{U,\mathcal{B}}(\sigma_U U_i, \sigma_U U_j) = \delta_{ij}.$$

We will show presently in Lemma 5.1 that B_U and $g_{U,\mathcal{B}}$ are independent of the particular normalized basis which was chosen, but this plays no role at present. If π is a linear subspace of \mathbb{R}^{3s} , set

$$\tilde{g}_\pi := \sigma_U^* g_{U,\mathcal{B}}|_\pi \quad \text{and} \quad \ell(\pi) := \text{Rank}\{\tilde{g}_\pi\}.$$

Let π be a spacelike k -plane where $k \geq 2$. Since every non-zero vector of π is spacelike, $\pi \cap \ker(\sigma_U) = \{0\}$. Thus $\ell(\pi) = k$. Let indices α, β range from 1 thru k ; let the index μ range from $k+1$ thru s . By diagonalizing the inner product \tilde{g}_π with respect to the positive definite inner product $g_{3s}|_\pi$, we can choose an orthonormal basis $\{X_\alpha\}$ for π and *positive constants* a_α so that

$$\tilde{g}_\pi(X_\alpha, X_\beta) = a_\alpha a_\beta \delta_{\alpha\beta} \quad \text{and} \quad g_{3s}(X_\alpha, X_\beta) = \delta_{\alpha\beta}.$$

By applying a symmetry of the form described in Equation (3.a), we may suppose

$$X_\alpha = a_\alpha U_\alpha + \sum_j \{b_{\alpha j} T_j + c_{\alpha j} V_j\} \quad \text{and} \quad g_{3s}(X_\alpha, X_\beta) = \delta_{\alpha\beta}$$

for suitably chosen constants $b_{\alpha j}$ and $c_{\alpha j}$. As $J_{3s}(\pi) = \sum_\alpha J_{3s}(X_\alpha)$, there exist constants $\varepsilon_{ij} = \varepsilon_{ij}(a, b, c) \in \mathbb{R}$ so

$$\begin{aligned} J_{3s}(\pi) : U_\beta &\rightarrow -\sum_{\alpha \neq \beta} a_\alpha^2 T_\beta + \sum_j \varepsilon_{\beta j} V_j, & J_{3s}(\pi) : T_\beta &\rightarrow \sum_{\alpha \neq \beta} a_\alpha^2 V_\beta, \\ J_{3s}(\pi) : U_\nu &\rightarrow -\sum_\alpha a_\alpha^2 T_\nu + \sum_j \varepsilon_{\nu j} V_j, & J_{3s}(\pi) : T_\nu &\rightarrow \sum_\alpha a_\alpha^2 V_\nu, \\ J_{3s}(\pi) : V_\beta &\rightarrow 0, & J_{3s}(\pi) : V_\nu &\rightarrow 0. \end{aligned}$$

Since $k \geq 2$, one has that $\sum_{\beta \neq \alpha} a_\beta^2 \neq 0$. Consequently

$$\begin{aligned} \text{Range } J_{3s}(\pi) &= \text{Span}\{T_1, \dots, T_s, V_1, \dots, V_s\}, \\ \text{Range } J_{3s}(\pi)^2 &= \text{Span}\{V_1, \dots, V_s\}, \quad \text{and} \quad \text{Range } J_{3s}(\pi)^3 = \{0\}. \end{aligned}$$

Assertion (1) now follows. Assertion (2) follows from Assertion (1).

Let π be a timelike k -plane. We diagonalize \tilde{g}_π with respect to the negative definite quadratic form $g_{3s}|_\pi$ to choose an orthonormal basis $\{X_\alpha\}$ for π so that

$$\tilde{g}_\pi(X_\alpha, X_\beta) = a_\alpha a_\beta \delta_{\alpha\beta} \quad \text{and} \quad g_{3s}(X_\alpha, X_\beta) = -\delta_{\alpha\beta}.$$

We have $\ell(\pi)$ is the number of times that $a_\alpha \neq 0$. Again, by applying an appropriate symmetry $\xi \in O(s)$ as described in Equation (3.a), we can assume without loss of generality

$$X_\alpha = a_\alpha U_\alpha + \sum_{1 \leq i \leq s} \{b_{\alpha i} T_i + c_{\alpha i} V_i\} \quad \text{for} \quad 1 \leq \alpha \leq k.$$

The calculations performed above show that if $\ell \geq 2$, then

$$(4.a) \quad \text{Rank}\{J_{3s}(\pi)\} = 2s, \quad \text{Rank}\{J_{3s}(\pi)^2\} = s, \quad \text{and} \quad J_{3s}(\pi)^3 = 0.$$

Since $\ker(\sigma_U) = \text{Span}\{T_i, V_i\}$, any timelike subspace of $\ker(\sigma_U)$ has dimension at most s . Since π is timelike, $\dim\{\pi \cap \ker(\sigma_U)\} \leq s$ and hence

$$\ell = \dim\{\sigma_U(\pi)\} = \dim\{\pi\} - \dim\{\pi \cap \ker(\sigma_U)\} \geq k - s.$$

Thus if $k \geq s + 2$, then $\ell \geq 2$. Assertion (3) now follows from Equation (4.a); Assertion (4) follows from Assertion (3).

To prove the final assertion, we must give examples of timelike k -planes whose Jacobi operators have different Jordan normal forms. The calculations performed above show that:

$$\text{Rank}\{J_{3s}(\pi)\} = \begin{cases} 0 & \text{if } \ell(\pi) = 0, \\ s-1 & \text{if } \ell(\pi) = 1, \\ s & \text{if } \ell(\pi) \geq 2. \end{cases}$$

If $2 \leq k \leq s+1$, set

$$\begin{aligned}\pi_1 &:= \begin{cases} \text{Span}\{T_1, \dots, T_k\} & \text{if } k \leq s, \\ \text{Span}\{T_1, \dots, T_s, Z_1^-\} & \text{if } k = s+1, \end{cases} \\ \pi_2 &:= \begin{cases} \text{Span}\{T_1, \dots, T_{k-1}, Z_1^-\} & \text{if } k \leq s, \\ \text{Span}\{T_1, \dots, T_{s-1}, Z_1^-, Z_2^-\} & \text{if } k = s+1. \end{cases}\end{aligned}$$

Then π_1 and π_2 are timelike k -planes with

$$\begin{aligned}\text{Rank}\{J_{3s}(\pi_1)\} &= \begin{cases} 0 & \text{if } k \leq s & \text{as } \ell(\pi) = 0, \\ s-1 & \text{if } k = s+1 & \text{as } \ell(\pi) = 1, \end{cases} \\ \neq \text{Rank}\{J_{3s}(\pi_2)\} &= \begin{cases} s-1 & \text{if } k \leq s & \text{as } \ell(\pi) = 1, \\ s & \text{if } k = s+1 & \text{as } \ell(\pi) = 2. \end{cases}\end{aligned}$$

Consequently \mathcal{V}_{3s} is not k -timelike higher order Jordan Osserman. \square

5. INVARIANTS OF THE MANIFOLD (M_F, g_F)

It is clear from the definition that $\|R\|_{g_F} = 0$ and $\|\nabla R\|_{g_F} = 0$. Thus to prove the final assertion of Theorem 1.3, we must introduce some additional structures and show that they are invariantly defined. We work on the model space \mathcal{V}_{3s} and suppress the index s in the interests of notational simplicity. We now return to structures introduced earlier in the proof of Lemma 4.1 and show these structures are intrinsic – i.e. they are independent of the particular normalized basis which was chosen. Consider the following subspaces of \mathbb{R}^{3s} :

$$\begin{aligned}A_V &:= \{W \in \mathbb{R}^{3s} : R(W_1, W_2, W_3, W) = 0 \ \forall \ W_1, W_2, W_3 \in \mathbb{R}^{3s}\}, \\ A_{T,V} &:= A_V^\perp = \{W \in \mathbb{R}^{3s} : g(W, W_1) = 0 \ \forall \ W_1 \in A_V\}.\end{aligned}$$

Let $\sigma_{U,T}$, σ_T , and σ_U be the natural projections to the quotient spaces

$$B_{U,T} := \mathbb{R}^{3s}/A_V, \quad B_T := A_{T,V}/A_V, \quad \text{and} \quad B_U := \mathbb{R}^{3s}/A_{T,V}.$$

The spaces given above are defined invariantly; they do not depend on the choice of basis. On the other hand, if $\mathcal{B} := \{U_i, T_i, V_i\}$ is **any** basis for \mathbb{R}^{3s} which satisfies the normalizations given in Definition 1.2, then one may express:

$$\begin{aligned}A_V &= \text{Span}\{V_i\}, & A_{T,V} &= \text{Span}\{T_i, V_i\}, \\ B_{U,T} &= \text{Span}\{\sigma_{U,T}U_i, \sigma_{U,T}T_i\}, & B_T &= \text{Span}\{\sigma_T T_i\}, \\ B_U &= \text{Span}\{\sigma_U U_i\}.\end{aligned}$$

The metric g_{3s} descends to a negative definite inner product g_T on $B_T \subset B_{U,T}$; $\{\sigma_T(T_i)\}$ is an orthonormal basis for B_T . Note that g_T is **not** defined on all of $B_{U,T}$ but only on the subspace B_T . Let $g_{U,\mathcal{B}}(\sigma_U U_i, \sigma_U U_j) = \delta_{ij}$ define a positive definite metric $g_{U,\mathcal{B}}$ on B_U which a priori depends on the basis \mathcal{B} .

Lemma 5.1. *We have $g_{U,\mathcal{B}} = g_{U,\tilde{\mathcal{B}}}$ for any two normalized bases \mathcal{B} and $\tilde{\mathcal{B}}$ of \mathbb{R}^{3s} .*

Proof. The tensor R descends to a tensor $R_{U,T}$ on $B_{U,T}$ so that $\sigma_{U,T}^* R_{U,T} = R$. The basis dependent action of the orthogonal group $O(s)$ on $T_P M$ described in Equation (3.a) induces basis dependent actions on the subspaces A_V and $A_{T,V}$ and on the quotient spaces B_U , B_T , and $B_{U,T}$ described above. This action preserves the metric g_T , the metric $g_{U,\mathcal{B}}$, and the tensor $R_{U,T}$.

Let \mathcal{B} and $\tilde{\mathcal{B}}$ be normalized bases for \mathbb{R}^3 . Then $\{\sigma_T(\tilde{T}_i)\}$ and $\{\sigma_T(T_i)\}$ are orthonormal bases for B_T . By replacing $\tilde{\mathcal{B}}$ by $\xi\tilde{\mathcal{B}}$ if necessary, where ξ is a suitably chosen element of $O(s)$, we can assume without loss of generality $\sigma_{U,T}\tilde{T}_i = \sigma_{U,T}T_i$ for all i . Let $u_i = \sigma_{U,T}U_i$, $\tilde{u}_i := \sigma_{U,T}\tilde{U}_i$, and $t_i := \sigma_{U,T}T_i = \sigma_{U,T}\tilde{T}_i$. Expand

$$\tilde{u}_j = \sum_{1 \leq k \leq s} \{a_{jk}u_k + b_{jk}t_k\}.$$

We shall prove the Lemma by showing that $a_{jk} = \delta_{jk}$.

Let $j \neq k$. We use the defining relations to see

$$\begin{aligned} 1 &= R_{U,T}(\tilde{u}_j, \tilde{u}_k, \tilde{u}_k, t_j) = (a_{jj}a_{kk} - a_{jk}a_{kj})a_{kk} \\ 0 &= R_{U,T}(\tilde{u}_j, \tilde{u}_k, \tilde{u}_j, t_j) = (a_{jj}a_{kk} - a_{jk}a_{kj})a_{jk}. \end{aligned}$$

Since $0 \neq (a_{jj}a_{kk} - a_{jk}a_{kj})$, we have $a_{jk} = 0$ for $j \neq k$; similarly $a_{kj} = 0$ for $j \neq k$. Thus $a_{jj}a_{kk}a_{kk} = 1$. Similarly $a_{jj}a_{kk}a_{jj} = 1$. Thus $a_{kk} = 1$ so $a_{jk} = \delta_{jk}$. \square

Proof of Theorem 1.3 (4). Fix $P \in TM$. Let $\{\partial_i^u, \partial_i^t, \partial_i^v\}$ be the coordinate frame for $T_P M$. We use the adjusted basis $\{U_i, T_i, V_i\}$ constructed in Section 2 to find an isomorphism Ψ which identifies $(T_P M, g_F, R_F)$ with \mathcal{V}_{3s} . As $\nabla R_F(\star, \star, \star, \star; \star) = 0$ if any entry belongs to $A_{T,V}$, there is a tensor ∇R_U on B_U so $\nabla R = \Psi^* \sigma_U^* \nabla R_U$. Let α_F be as defined previously. We use Remark 2.2 to see

$$\begin{aligned} \alpha_P &= \frac{1}{4} \sum_{i_1, i_2, i_3, i_4, i_5} \nabla R(\partial_{i_1}^u, \partial_{i_2}^u, \partial_{i_3}^u, \partial_{i_4}^u, \partial_{i_5}^u)^2 \\ &= \frac{1}{4} \sum_{i_1, i_2, i_3, i_4, i_5} \nabla R(U_{i_1}, U_{i_2}, U_{i_3}, U_{i_4}, U_{i_5})^2 = \frac{1}{4} \|\nabla R_U\|_{g_U}^2. \end{aligned}$$

As $\|\nabla R_U\|_{g_U}^2$ is invariantly defined, α_P is preserved by local isometries. Thus, if (M_F, g_F) is locally homogeneous, then α_P is constant. This fails for generic F . \square

Remark 5.2. We can construct additional invariants of the metric by considering the norms of higher order covariant derivatives of the curvature tensor. Set:

$$\begin{aligned} \alpha_F^k : &= 2^{-k-1} \|\nabla^{(k)} R\|_{g_U} \\ &= 2^{-k-1} \sum_{i_1, i_2, i_3, i_4, j_1, \dots, j_k} R(\partial_{i_1}^u, \partial_{i_2}^u, \partial_{i_3}^u, \partial_{i_4}^u; \partial_{j_1}^u, \dots, \partial_{j_k}^u)^2. \end{aligned}$$

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